ISSN: 2456–5474 RNI No.UPBIL/2016/68367 Vol.-5* Issue-12* January- 2021
Innovation The Research Concept

# A Study on Algebraic Geometry

Paper Submission: 15/01/2021, Date of Acceptance: 26/01/2021, Date of Publication: 27/01/2021

**Abstract**

Algebraic geometry mainly deals with the different types of curves or surfaces that can be easily viewed as both geometrical objects and algebraically equations. One of the main objectives of the analysis is to introduce the concepts of algebraic geometry and to investigate the issue related with the algebraic geometry. The review paper also described the objects of algorithm and algebra.

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**Keywords:** Geometry, Equations, Algebra.

**Introduction**

Curves of algebra are considered as one of the oldest and most analyzed branches of algebraic geometry and in fact it is considered as one of the most investigated subject in the field of mathematics. As they gives some of the most thrilling issues of classical mathematics. However, they have determine some applications in various regions of science and technology like computer vision, coding theory, information regarding to quantum theory, biomathematics etc. Nonetheless, surprisingly sufficient, there are some fundamental queries related to algebraic curves of theories in details which yet avoid the groups of specialist in this field of investigation. We deeply explain some issues of interest which can be analyzed any using some computational methods or techniques.

**Review of Literature**

One of the main branch of modern mathematics is Algebraic geometry, with roots from classical Italian geometries. The modern developed flavor begun with Grothendieckand persisted with several explanations related to algebraic geometers of the second half of the of the 20th century . At the time of last 20 years, the basic subject has been changed tremendously because of growth of modern evaluation process and access to better evaluation of power. These types of changes have encouraged a modern direction of algebraic geometry which is also called computational algebraic geometry. It basically comprised the field of algebraic geometry where computer algebra can be utilized to obtain explicit outcomes. It is perfectly belongs to such area will be of deep influence and significance in the future mathematics field. Moreover, these modern growth have made possible applications of algebraic geometry in such areas as a coding theory, computer security, computer vision and so on [2].

As per the theory of [9], The growth of the computation al process or methods in the past ten years has made easy to attack on several classical query of algebraic geometry from a overview of computational techniques. The investigation explains some open queries of computational algebraic geometry which can be advanced from such overviews. Most of the discussions on queries are basically the decomposition of Jacobians of genus of 2 curves, auto morphism groups of algebraic curves and the corresponding loci in the moduli space of algebraic curves Mg with the functioning curves of auto morphism etc.

**Objective of the Study**

1. To investigate the basic and fundamental concepts of algebraic geometry.
2. To research about the usefulness of algebra to mathematics investigation.
3. To investigate the issue related to the algebraic geometry

**Algebraic Geomtery Objects**

The fundamental issues of algebraic geometry is to comprehend the set of points $x = (x_1 \dots\dots\dots\dots\dots\dots\dots . x_n) \in K^n$ satisfying a system of equations.

$$f_1(x_1 \dots\dots\dots\dots . x_n) = 0$$
$$f_k(x_1 \dots\dots\dots\dots . x_n) = 0$$

Where K is a field and $f_1 \dots\dots\dots\dots\dots\dots\dots . f_k$ are elements of the polynomial ring $K[x] = K[x_1 \dots\dots\dots x_n]$ .

The solution set of $f_1 = 0, \dots\dots\dots . f_k = 0$ is called the algebraic set, or algebraic varietyof $f_1 \dots\dots\dots\dots\dots\dots\dots . f_k$ and is denoted by

$$V = V(f_1 \dots\dots\dots\dots\dots\dots\dots . f_k)$$

It is easy to see, and important to know, that Vdepends only on the ideal

I= $\{f_1 \dots\dots . f_k\} = \{f \in K[x] | \ f = \sum_{i=1}^{k} a_i f_i, a_i \in K[x]\}$

Generated by $f_1 \dots\dots . f_k$ in K[x],

That is V= V (I) = $\{ x \in K^n \mid f(x) = 0 \forall f \in I\}$

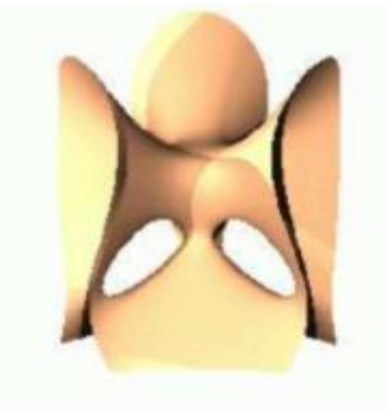


**Figure: The Clebsch Cubic**

This given equation is of completely uniquecubic surface which has $S_5$, which refer to the symmetry group of 5 letters, as symmetry group. Alfred Clebsch discover it and it has the affine expression such as:

$$81(x^3 + y^3 + z^3) - 189(x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2) + 54xyz + 126(xy + xz + yz) - 9(x^2 + y^2 + z^2) - 9(x + y + z) + 1 = 0$$

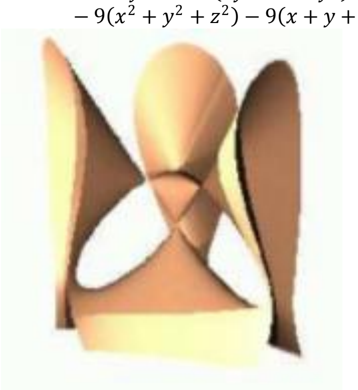


**Figure: The Cayley Cubic**

The given figure is of completely unique cubic surface which has about 4 ordinary double points, generally called as Cayley cubic because it was discover by Arthur Cayley. It is also considered as the degeneration of the Clebsch cubic which has basically S4 as symmetry group and the projective equation is expressed as:

$$z_0z_1z_2 + z_0z_1z_3 + z_0z_2z_3 + z_1z_2z_3 = 0$$

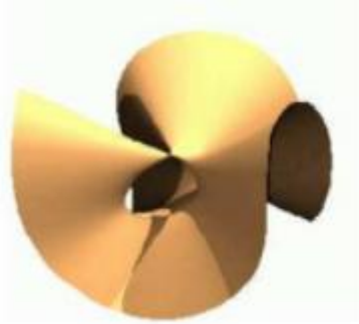


**Figure: A Cubic with a $D_4$- Singularity**

A Cubic with a D4-Singularity degenerating the Cayley cubic we receiveaD4-singularity. The affine equation is basically represented with expression:

$$x(x^2 - y^2) + z^2(1 + z) + \frac{2}{5}xy + \frac{2}{5}yz = 0$$

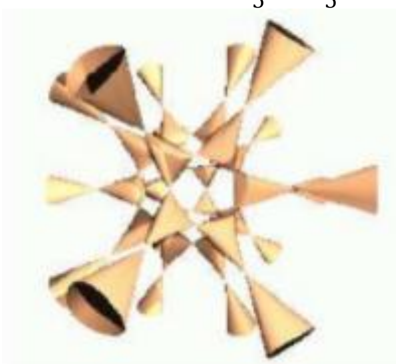


**Figure: The Barth Sextie**

The expression for this sextic was mainly found by the great mathematician Wolf Barth. It has generally 65 ordinary double points, the maximal probable number for a sextic. Its affine expression basically represented as:

with $c = \frac{1+\sqrt{5}}{2}$)

$$(8c + 4)x^2y^2z^2 - c^4(x^4y^2 + y^4z^2 + x^2z^4) + c^2(x^2y^4 + y^2z^4 + x^4z^2) - 2c + 14(x^2 + y^2 + z^2 - 1)^2 = 0$$

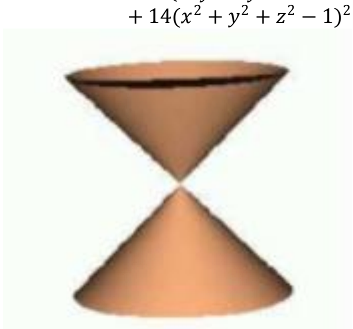
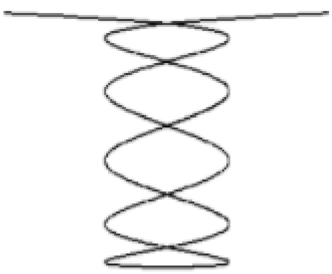


**Figure: An Ordinary Node**

An ordinary node is the simplest sin-gularity. It has the local equation.$x^2 + y^2 - z^2 = 0$

**Figure: Whitney's Umbrella**

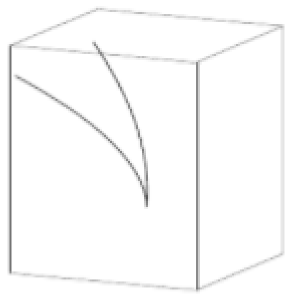
The above Whitney umbrella is mainly named because it was discover by Hassler Whitney who basically studied it in connection with the proper satisfaction of studied spaces. It's generally expressed by the equation as: $x^2y^2 - zx^2 = 0$



**A 5-nodal plane curve of degree 11with equation**

$$-16x^2 + 1048576y^{11} - 720896y^9 + 180224y^7 - 19712y^5 + 880y^3 - 11y + 12,$$

a deformation of $A_{10}: y^{11} - x^2 = 0$



This space curve is given parametricallyby $x = t^4, y = t^3, z = t^2$,

or implicitly by $x - z^2 = y^2 - z^3 = 0$

Of course, if for some polynomial$f \in K[x]f^d \,|v = 0$then $f|v = 0$and hence, V=V(I)depends only on the radical of I,

$$\sqrt{I} = \{f \in K[x]f^d \in I, for\ some\ d\}$$

The biggest ideal determined by Vis

$$I(V) = \{f \in K[x]|f(x) = 0\ \forall\ x\ \in V\}$$

and we have$I \subset \sqrt{I} \subset I(V) and\ V(I(V)) = V(\sqrt{I}) = V(I) = V$

The important Hilbert Nullstellensatz states that, for Kan algebraically closed field, wehave for any variety $V \subset K^n$and any ideal$J \subset K[x]$,

$$V = V(J) \Rightarrow I(V) = \sqrt{J}$$

(The converse implication being trivial). That is, we can recover the ideal J, up to radical, justfrom its zero set and, therefore, for fields like C (but, unfortunately, not for R) geometry andalgebra are "almost equal". But almost equal is not equal and we shall have occasion to see thatthe diﬀerence between Iand √I has very visible geometric consequences.

The above image perfectly explains the beauty of algebraic geometric objects and variations which had some noticeable impacts in the growth of algebraic geometry and singularity of object.

The above expression of Clebsch cubic has been the object of large number of study in worldwide algebraic geometry, the D4 and the Cayley cubic also but furthermore, ever since the D4 cubic deforms through the Cayley cubic, to the Clebsch cubic, the first 3 images perfectly explains the deformation hypothesis, an essential branch or network of algebraic geometry.

The general node which is also called as A1-singularity, it is the basically considered as one of the simple singularity in any of the dimension.

The above figure of Barth sextic perfectly explains a fundamental expression but very difficult and yet unsolved an issue: to find out the maximum probable several number of singularities on a projective variations of given degree.

At the starting of stratification theory, Whitney's umbrella, an essential example for the two Whitney conditions as follows:

The 5-nodal plane curve perfectly describe the worldwide survival issues.

Furthermore, all these types of deformations with the highest number of nodes play also a perfect role in the domestic hypothesis of singularities expression.

Algebraic Constructs

**Rings and Fields**

A ring is a set A together with two operations, say '+' and '.' (called addition and multiplication) which satisfy the following:

(Rl) a + b = b + a for all a, bE A (commutativity of +)

(R2) a + (b + c) =. (a + b) + c; a.(b.c) = (a.b).c for all a, b, c E A (associativity of + and .)

(R3) There exists an element 0 satisfying a + 0 = a for all a E A (existence of additive identity).

(R4) For each a E A, there exists an element ( -a) such that a + ( -a) = 0 (existence of additive inverses).

(RS) a.(b + c) = a.b + a.c, and (b + c).a = b.a + c.a for all a, b, c E A (left and right distributivity).

If in addition, we also have a.b = b.a for all a, b E A, we call it a commutative ring. If there exists an element 1 E A, with 1 f:. 0 and satisfying a.l = a = l.a for all a E A, we call it a ring with identity. A subring B of a ring A is a subset of A which is also a ring with the + and operations from A. If A is a commutative ring with identity such that for each non-zero element a E A, there exists an element a-I (called the multiplicative inverse of a) satisfying a.a-1 = 1, we call A a field. So a field is an abelian group with respect to

(+), and its non-zero elements form an abelian group with respect to multiplication '.'.

Fields are usually denoted by the lowercase letter k, or the bold uppercase F. If a ring A is a vector space! Over a I Take the definition of an Rfield k, such that the scalar multiplication from k is com- vector space, replace R by k patible with the ring operations (Le. A(a + b) = Aa + Ab everywhere, and you have the and A(a.b) = (Aa).b = a.(Ab) for all A E k and all. a, b E A) definition of a k-vector space. then A is called a k-algebra. In the particular case when a ring A contains a field k as a subring, A clearly becomes a k-algebra, with vector addition being the ring addition +, and scalar multiplication coming from ring multiplication'.' by elements of k.

**Examples of Rings, Fields, Algebras**

1. The set of integers Z is a commutative ring with identity, with its usual addition and multiplication operations. The rational numbers Q, the real numbers R and complex num~ers C with their usual operations are fields. Clearly Z C Q eRe C is a chain in which each inclusion is one of a subring in a bigger ring. Thus R is a Q-algebra (of infinite, in fact uncountable dimension as a Q vector space), and C is an R-algebra of vector space dimension 2 (spanned by the basis {I, i}).
2. Let m 2: 2 be a natural number. The set of integers modulo m, (also called residue classes mod m) denoted Zm, is the set {a, I, .. , m - I}. The sum a + b is defined as if where r is the remainder « m) upon dividing the integer a+b by m. Multiplication is defined similarly. These are called addition and multiplication modulo m, and Zm becomes a commutative ring with identity under these operations. The properties R1 through R5 for Zm follow from the corresponding properties for Z.
   If m = p a prime, then Zp becomes a field (why?), often denoted F p to emphasise its 'fieldhood'.
3. The set of continuous complex valued functions on the closed interval (0,1] is a commutative ring with identity under the operations of multiplying and adding continuous functions pointwise, and is denoted C([O, 1]). Likewise C(R), the commutative ring of continuous complex valued functions on R. The ring C(R) contains the subring Cc(R) of continuous functions on R of compact support (Le. continuous functions I such that I(x) = 0 for Ixl 2: a, for some a depending on I). Note Cc(R) is a ring without identity, (the constant function 1 is not of compact support!). In fact C(R), since it contains the constant functions, is a C-algebra. Cc(R) is also C-algebra (because multiplying a continuous compactly supported function with A E C gives a continuous compactly supported function) even though it does not contain C as a subalgebra! Another interesting way of making Cc(R) a ring is to retain the old addition, but make multipli- cation the convolution product 1* 9 defined by:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\mathrm{d}y.$$

4. The set of n x n matrices with entries in a commutative ring A with identity, is a ring with identity under matrix addition and mUltiplication. It is denoted by M (n, A), and is not a commutative ring (therefore not a field) if n 2: 2. More generally, for those who are aware of Hilbert spaces, the set of all bounded operators B(1£) on a Hilbert space 1£ is a ring with identity.

**Ideals**

From this point on, all rings we consider will be assumed to be commutative, unless otherwise stated. For the sake of convenience we will write ab instead of a.b for the product of the elements a and b in a ring. An ideal I of a ring A is a subset of A such that: (II) a + bEl for all a, bEl. (I2) ax E I for all a E A, x E I. Clearly, if I contains the identity element 1, the property (I2) of an ideal would force I to be all of A. Thus the interesting ideals do not contain 1 or for that matter, any invertible elements. An ideal I in a ring A which is not equal to A is called a proper ideal.

**Fundamental Theorem of Algebra**

Rach and every degree of polynomial ≥ 1 in C[X] have a root in C. This expression of theorem is the crucial ley to why all complex number shave some of the magical powers. An evident of this outcome can be attained by using the Liouville theorem in complicated analysis or some elementary topology.

Corollary: Each and every polynomial of degree ≥ 1 in C[X] is a basic product of polynomial of degree 1, i.e. linear polynomials.

Proof: If deg I(X) = 1, there is nothing to prove. Otherwise, note that f has a root a E C, and is therefore divisible by (X - a) (prove!). Thus f = (X - a)g(X) where deg g = deg f-1. Induct on degree.

Thus the only irreducible polynomials in C[X] which are of degree ≥1 are linear polynomials

**Problems in Algebraic Geometry**

The one of teg most basic and most essential application of Grobner bases to algebraic constructions. Since these can be determine in more or less any network dealing with Grobner bases:

- Ideal (resp. module) membership problem
- Intersection with subrings (elimination of variables)
- Intersection of ideals (resp. submodules)
- Zariski closure of the image of a map
- Solvability of polynomial equations
- Solving polynomial equations
- Radical membership
- Quotient of ideals
- Saturation of ideals
- Kernel of a module homomorphism
- Kernel of a ring homomorphism
- Algebraic relations between polynomials
- Hilbert polynomial of graded ideals and modules

The further queries and issues lead to algorithms which are highly more and more involved. They are, however, still very basic and simple natural.

This can be easily explains by the way of 4 simple and natural examples, representing in the pictures, referred to as example 1)-4):

Assume, we are given an ideal $I \subset K(x_1 \dots\dots\dots . x_n)$ by a set of generators $(f_1 \dots\dots\dots . f_k) \in K[x]$ .Consider the following questions and problems:

1 Is V (I) irreducible or may it be decomposed into several algebraic varieties? If so, find its irreducible components.

In case of algebraically, it means to evaluate a primary or initial decomposition of I or of √I, the latter means to compute the associated prime ideals of I.

Example 1) is irreducible, Example 2) has two components (one of dimension 2 and oneof dimension 1), Example 3) has three (one–dimensional) and Example 4) has nine (zero–dimensional) components.

2. Is I a radical ideal (that is, I=√I)? If not, compute its radical √I

In Examples 1) – 3) I is radical while in Example 4)$\sqrt{I} = (y^3 - y, x^3 - x)$ which is muchsimpler than I. In this example the central point corresponds to $V((x,y)^2)$ which is a fatpoint, that is, it is a solution of Iof multiplicity $(= dim_k K[x,y]/(x,y^2)$ bigger than 1 (equal to 3). All other points have multiplicity 1, hence the total number of solutions (counted with multiplicity) is 11.

3. A natural question to ask is "how independent are the generators $(f_1 \dots\dots\dots . f_k) of\ I?$, that is, we ask for all relations

$(r_1 \dots\dots\dots . r_k \in K(x^k)$ such that $\sum r_i f_i = 0$ .

These relations form a submodule of $K(x^k)$, which is called the syzygy module of $f_1 \dots\dots\dots . f_k$ and is denoted by syz (I).

It is the kernel of the K[x]–linear map

$K(x^k) \to k[x], ; r_1 \dots\dots\dots . r_k \leftrightarrow \sum r_i f_i$

4. In general terms, Generators of the kernel of a K[x]- linear map $K(x)^r \to K(x)^x$ or alternatively, for solutions of a system of linear expression over K[x]. A complete direct geometric interpretation of syzygies is not transparent, but there are some instances where characteristic of syzygies have essential for the geometric outcomes.

More generally, generators of the kernel of a K[x]–linear map $K(x)^r \to K(x)^x$ or, in other words, for solutions of a system of linear equations over K[x].A direct geometric interpretation of syzygies is not so clear, but there are instances where properties of syzygies have important geometric consequences.

5. Different issues is related to the Riemann singularity removable hypothesis which basically states that a function on a complicated manifold that is mainly holomorphic and bounded outside a sub-variety of co-dimension 1, which is actually holomorphic everywhere. This is well known for open subsets of C, but in higher dimension there exists a second singularity removable theorem, which states that a function, which is holomorphic outside a sub-variety of co-dimension 2(no assumption on boundedness), is holomorphic everywhere. For singular complex varieties this is not true in general, but those for which the two removable theorems hold are called normal. Moreover, each reduced variety has a normalisation and there is morphism with finite fibres from the normalisation to the variety, which is an isomorphism outside the singular locus.

5. The importance of singularities looks like to not only in the normalization issue. The investigation of singularities is also called domestic or local algebraic geometric and also belongs to the fundamental tasks of algebraic geometry. Presently, Theory of singularity is an entire subject on its own. A variety of singularity theory is a point which has no neighborhood in which the Jacobian matrix of the generation has persistent and constant rank.

The issue related to geometry ought to be interested as properties of the variety in a neighborhood of the origin or more normally, the given point or expression.

**Some Global Algorithms**

Here some issues or queries mentioned related to geometry, 2 algorithms are basically related to all these issues:

1. Primary decomposition
2. Normalization

**Primary Algorithm Decomposition**

All essential algorithms for primary decompositions in K[x] are simply involved and use many types of sub-algorithms from different part of computer algebra in specific Grobner bases, properties, sets and multiplicative polynomial factorization over some broad range of the field K. For an effective execution which can treat examples of interest in algebraic geometry, several additional small algorithms have to be used.

There are some of the major ingredients for primary decomposition as follows:

1. Reduction to zero–dimensional primary decomposition (GTZ); maximal independent sets; ideal quotient, saturation, intersection.
2. Zero–dimensional primary decomposition (GTZ); lexicographical Gr¨obner basis; factorisation of multivariate polynomials; generic change of variables; primitive element computation.

**Related Algorithms**

1. Computation of the radical; square–free part of univariate polynomials; find (random) regular sequences (EHV).
2. Computation of the equidimensional part (EHV); Ext–annihilators; ideal quotients, saturation and intersection.

**Normalisation**

Different essential algorithm is the normalization of K[x]/I where I refer to the radical ideal. This can be idealy used as step in the primary decomposition as previously stated in [1] but also their interest. Various algorithms have been stated mainly by [3][10][11]. As per the theory of [4], a fundamental constructive evident for the ideal of the non-normal locus of a complicated space. Within this evidence, they also gives a criteria of normality which is significantly an algorithm for evaluating the normalization. As per the explanation De Jong (1998), to make the algorithm more effectively required additional work. As per the theory of Decker, Greuel, De Jong and Pfister (1998, the Grauert Remmert algorithm is mainly imposed in SINGULAR and looks to be only complete execution of the normalization.

Here are some of the ingredients which is essential for the normalization:

1. Computation of the ideal Jove the singular locus of the ideal I;
2. Computation of a non–zero divisor for J;
3. Ring structure on Hom(J, J);
4. Syzygies, normal forms, ideal quotient

In the preceding picture, R, the normalisation of S, is just the polynomial ring in two variablesT(1) and T(2). (The "handle" of Whitney's umbrella is invisible in the parametric picture sinceit requires an imaginary parameter t.)

**Conclusion**

This investigation basically exhibit that all the queries related to the algorithm needs computational solutions which mainly used at some place as Grobner basis explained. Furthermore, there are several algorithms of algebra which mainly plays a major role in contribution to mathematical investigation.

**References**


1. *Eisenbud, D.; Huneke, C.; Vasconcelos, W. (1992). Direct methods for primary decomposition. Invent.Math. 110, 207–235*
2. *Elezi and Shaska,2007 SPECIAL ISSUE ON ALGEBRA ANDCOMPUTATIONAL ALGEBRAIC GEOMETRY*
3. *Gianni, P.; Trager, B. (1997). Integral closure of noetherian rings. Proceedings ISSAC 97*
4. *Grauert, H.; Remmert, R. (1971). Analytische Stellenalgebren. Springer Verlag*
5. *Gray (2018) Algebraic Number Theory: Cyclotomy. In: A History of Abstract Algebra. Springer Undergraduate Mathematics Series. Springer, Cham.*
6. *L Ahlfors, ComplexAnalysis, 3rd Ed., McGraw Hill.*
7. *M Atiyah and I G MacDonald, Introduction to Commutative Algebra, AddisonWesley, 1969.*
8. *N Jacobson, Basic Algebra, Vol. 1, Freeman, 1974.*
9. *Shaska(2018) SOME OPEN PROBLEMS IN COMPUTATIONAL ALGEBRAIC GEOMETRY ALBANIAN JOURNAL OF MATHEMATICS Volume 1, Number 4, Pages 297–319 ISSN 1930-1235: (2018)*
10. *Stolzenberg, G. (1968). Constructive normalization of an algebraic variety. Bull. Amer. Math. Soc. 74,595–599*
11. *Vasconcelos, V.W. (1991). Computing the integral closure of an affine domain. Proc. AMS 113 (3),633–638*